# Prediction of Elastic-Airplane Lateral Dynamics from Rigid-Body Aerodynamics

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**Control-configured vehicle technology has increased the demand for detailed analysis of dynamic stability and control, handling and ride qualities, and control system dynamics at the early stages of design and development. For these early analyses an approximate, but reasonably accurate, set of equations of motion for elastic airplanes is needed. Such a formulation is developed for the lateral dynamics of elastic airplanes. It makes use of rigid-body aerodynamic stability derivatives and the antisymmetric elastic mode shapes and frequencies in formulating the forces and moments due to elastic motion. Verification of accuracy was made by comparison with B-1 airplane dynamics obtained by other methods. With four antisymmetric elastic modes included, frequencies and damping ratios of the coupled modes agree reasonably well.**

## Nomenclature

$b$ = wing span, ft
$c$ = mean aerodynamic wing chord, ft
c.g. = center of gravity
$C_{L_\alpha}$ = lift curve slope
$C_{l_p}$ = rolling moment coefficient due to roll rate
$C_{l_r}$ = rolling moment coefficient due to yaw rate
$C_{l_\beta}$ = rolling moment coefficient due to sideslip angle
$C_{l_{\delta_a}}$ = rolling moment coefficient due to aileron deflection
$C_{n_p}$ = yawing moment coefficient due to roll rate
$C_{n_r}$ = yawing moment coefficient due to yaw rate
$C_{n_\beta}$ = yawing moment coefficient due to sideslip angle
$C_{y_p}$ = side force coefficient due to roll rate
$C_{y_r}$ = side force coefficient due to yaw rate
$C_{y_\beta}$ = side force coefficient due to sideslip
$C_{y_{\delta_r}}$ = side force coefficient due to rudder deflection
$I_{xz}$ = product of inertia, slug-ft$^2$
$I_x$ = mass moment of inertia about $x$ axis, slug-ft$^2$
$I_z$ = mass moment of inertia about $z$ axis, slug-ft$^2$
$M$ = total airplane mass, slugs
$\mathfrak{M}_i$ = $i$th elastic mode generalized mass, slugs
$S$ = lifting surface reference area, ft$^2$
$U_0$ = trim flight velocity, ft/s
$Z_w$ = aerodynamic force stability derivative due to plunge velocity of c.g., 1/s
$\delta_a$ = aileron deflection, rad
$\delta_r$ = rudder deflection, rad
$\zeta_i$ = $i$th elastic mode structural damping ratio
$\xi_i$ = $i$th elastic mode generalized displacement, ft
$\dot{\xi}_i$ = $i$th elastic mode generalized velocity, ft/s
$\rho$ = freestream air density, slugs/ft$^3$
$\omega_i$ = undamped natural frequency of $i$th free-free elastic mode, 1/s
$\phi_i(x,y)$ = $i$th elastic mode normalized mode shape in $xy$ plane
$\phi_i(x,z)$ = $i$th elastic mode normalized mode shape in $xz$ plane
$\phi_i'(x,y)$ = slope of $\phi_i(x,y)$ with respect to $x$, 1/ft
$\phi_i'(x,z)$ = slope of $\phi_i(x,z)$ with respect to $x$, 1/ft
$'\phi_i(x,y)$ = slope of $\phi_i(x,y)$ with respect to $y$, 1/ft
$\Xi_{im}(t)$ = $i$th elastic mode motion-dependent generalized force, ft/s$^2$
$\Xi_{ig}(t)$ = $i$th elastic mode gust-induced generalized force, ft/s$^2$

*Subscripts*

$W$ = wing
$HT$ = horizontal tail
$VT$ = vertical tail
$f$ = fuselage

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## Introduction

RECENT work with control-configured vehicle (CCV) and active control technology (ACT) has improved the performance, stability, and handling qualities of flexible airplanes and has opened up a new realm of aircraft design frontiers.[1] The present trend in aircraft design is geared toward larger airplanes and the use of lighter and more flexible materials. Therefore, the elastic behavior of these vehicles has an appreciable influence on their handling and ride qualities. The potentially adverse effects create a definite need for a simplified method of modeling aeroelastic dynamics during the preliminary stage of control system design and development for new airplanes.

The rigid-body aerodynamic stability derivatives for preliminary design are usually available only as calculated values from sources such as DATCOM.[2] These provide little, if any, information on the stability derivatives due to elastic modes. However, calculated values of the symmetric and antisymmetric orthogonal elastic vibration mode shapes and natural frequencies are usually available for use in equations of motion formulation.

We have developed a formulation of the lateral-directional equations of motion for elastic airplanes that makes use of rigid-body aerodynamic stability derivatives and the elastic mode shapes and frequencies to describe the aerodynamic forces and moments due to the elastic motion of the aircraft. The B-1 airplane is used as a model for accuracy verification. A formulation for longitudinal dynamics is presented in Ref. 3.

## Equations of Motion Formulation

The formulation of the small-perturbation aerodynamic forces and moments is based on the local effective sideslip

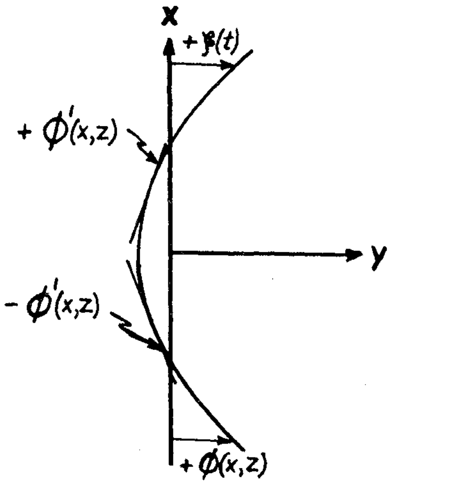


Fig. 1 $xz$ plane sign convention.

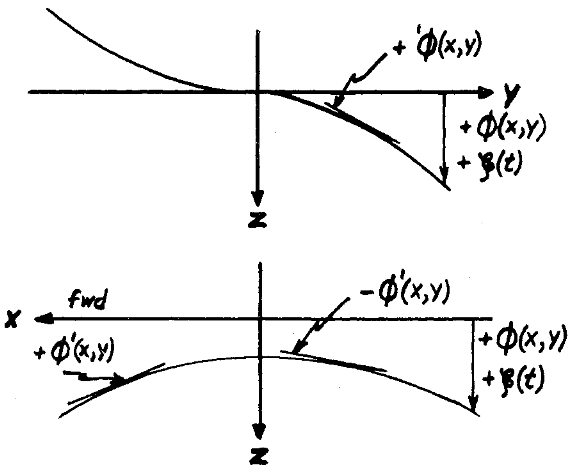


Fig. 2 $xy$ plane sign convention.

angle $\beta(x,z,t)$, or the effective sideslip velocity $v(x,z,t)$, where $v(x,z,t) = U_0\beta(x,z,t)$, the local effective roll rate $p(x,y,t)$, the local effective yaw rate $r(x,z,t)$, and the local antisymmetric effective plunge velocity $w(x,y,t)$. This is analogous to piston theory.

The elastic vibration characteristics are based on the usual idealization of the structure as two flat plates, one in the $xy$ plane, the area of which consists of the wing, fuselage, and horizontal tail planform areas, and the other in the $xz$ plane, the area of which consists of the fuselage and vertical tail profile areas. The antisymmetrical orthogonal free-free elastic vibration mode shapes, $\phi_i(x,y)$ and $\phi_i(x,z)$, are functions of the $x$, $y$, and $z$ coordinates in the body axes system originating at the center of gravity.[4] The sign convention for the mode shapes, mode slopes, and generalized displacements is shown in Figs. 1 and 2.

The three rigid-body and $n$ elastic mode small perturbation equations of motion about a trim condition are given by Eq. (1). $v(x,y,t)$, $p(x,z,t)$, and $r(x,y,t)$ are considered zero:

$$\dot{v}(t) - g\phi(t) - U_0\dot{\psi}(t) = \int_z\int_x \frac{\partial^2 Y_v}{\partial x \partial z} v(x,z,t)\,dxdz + \int_y\int_x \frac{\partial^2 Y_p}{\partial x \partial y} p(x,y,t)\,dxdy + \int_z\int_x \frac{\partial^2 Y_r}{\partial x \partial z} r(x,z,t)\,dxdz + Y_{\delta_r}\delta_r(t) + Y_v v_g(t) + Y_r r_g(t) + Y_p p_g(t) \quad (1a)$$

$$\ddot{\phi}(t) - \frac{I_{xz}}{I_x}\ddot{\psi}(t) = \int_z\int_x \frac{\partial^2 L_v}{\partial x \partial z} v(x,z,t)\,dxdz + \int_y\int_x \frac{\partial^2 L_p}{\partial x \partial y} p(x,y,t)\,dxdy + \int_z\int_x \frac{\partial^2 L_r}{\partial x \partial z} r(x,z,t)\,dxdz + \frac{M}{I_x}\int_y\int_x y\frac{\partial^2 Z_w}{\partial x \partial y} w(x,y,t)\,dxdy + L_{\delta_a}\delta_a(t) + L_{\delta_r}\delta_r(t) + L_v v_g(t) + L_p p_g(t) + L_r r_g(t) \quad (1b)$$

$$\ddot{\psi}(t) - \frac{I_{xz}}{I_z}\ddot{\phi}(t) = \int_z\int_x \frac{\partial^2 N_v}{\partial x \partial z} v(x,z,t)\,dxdz + \int_y\int_x \frac{\partial^2 N_p}{\partial x \partial y} p(x,y,t)\,dxdy + \int_z\int_x \frac{\partial^2 N_r}{\partial x \partial z} (x,z,t)\,dxdz + N_{\delta_a}\delta_a(t) + N_{\delta_r}\delta_r(t) + N_v v_g(t) + N_p p_g(t) + N_r r_g(t) \quad (1c)$$

$$\ddot{\xi}_i(t) + 2\zeta_i\omega_i\dot{\xi}_i(t) + \omega_i^2\xi_i(t) = \Xi_{im}(t) + \Xi_{ig}(t) \quad (i=1,2,\ldots,n) \quad (1d)$$

$Y_p$, $Y_v$, $Y_r$, $Y_{\delta_r}$, $L_v$, $L_p$, $L_r$, $L_{\delta_a}$, $L_{\delta_r}$, $Z_w$, $N_v$, $N_p$, $N_r$, $N_{\delta_a}N_{\delta_r}$ are the rigid-body dimensional aerodynamic stability derivatives defined in Ref. 5. For example,

$$N_v = \rho U_0 S b C_{n_\beta}/2I_z \quad (2)$$

All the $\dot{v}$ stability derivatives and $Y_{\delta_a}$ are assumed to be negligible and are not included in Eqs. (1a-1d). Also, $\partial^2 Y_p/\partial x \partial y$ in Eq. (1a) is considered zero, since wing rolling contributes very little to side force. Implicit in the intergrals of Eq. (1) is the assumption that aeroelastic deformation does not significantly change the stability derivates from rigid-body values. Residual stiffness or residual flexibility corrections to the stability derivatives are neglected. The aerodynamics are assumed quasisteady; i.e., only the zero frequency terms are retained.

The integral terms and the generalized force terms in Eqs. (1a-1d) are functions of $v(x,z,t)$, $p(x,y,t)$, $r(x,z,t)$, and $w(x,y,t)$, which by use of the sign convention in Figs. 1 and 2, can be closely approximated by Eqs. (3-6).

$$v(x,z,t) = v(t) + \sum_{i=1}^{n}[\phi_i(x,z)\dot{\xi}_i(t) - U_0\phi_i'(x,z)\xi_i(t)] \quad (3)$$

$$p(x,y,t) = p(t) + \sum_{i=1}^{n}{}'\phi_i(x,y)\dot{\xi}_i(t) \quad (4)$$

$$r(x,z,t) = r(t) + \sum_{i=1}^{n}\phi_i'(x,z)\dot{\xi}_i(t) \quad (5)$$

$$w(x,y,t) = \sum_{i=1}^{n}[\phi_i(x,y)\dot{\xi}_i(t) - U_0\phi_i'(x,y)\xi_i(t)] \quad (6)$$

By substituting Eqs. (3-6) in Eq. (1), the integral terms can be expressed as in Eqs. (7-9).

$$\int_z\int_x \frac{\partial^2 Y_v}{\partial x \partial z} v(x,z,t)\,dxdz + \int_z\int_x \frac{\partial^2 Y_r}{\partial x \partial z} r(x,z,t)\,dxdz + \int_y\int_x \frac{\partial^2 Y_p}{\partial x \partial y} p(x,y,t)\,dxdy = Y_p\dot{\phi} + Y_r\dot{\psi}(t) + Y_v v(t) + \sum_{i=1}^{n}[Y_{\xi_i}\xi_i(t) + Y_{\dot{\xi}_i}\dot{\xi}_i(t)] \quad (7)$$

$$\int_z \int_x \frac{\partial^2 L_v}{\partial x \partial z} v(x,z,t)\,dxdz + \int_y \int_x \frac{\partial^2 L_p}{\partial x \partial y} p(x,y,t)\,dxdy$$
$$\int_z \int_x \frac{\partial^2 L_r}{\partial x \partial z} r(x,z,t)\,dxdz + \frac{M}{I_x}\int_y \int_x y\frac{\partial^2 Z_w}{\partial x \partial y} w(x,y,t)\,dxdy$$
$$= L_v v(t) + L_p \dot\phi(t) + L_r \dot\psi(t) + \sum_{i=1}^{n} [L_{\xi_i}\xi_i(t) + L_{\dot\xi_i}\dot\xi_i(t)] \quad (8)$$

$$\int_z \int_x \frac{\partial^2 N_v}{\partial x \partial z} v(x,z,t)\,dxdz + \int_y \int_x \frac{\partial^2 N_p}{\partial x \partial y} p(x,y,t)\,dxdy$$
$$+ \int_z \int_x \frac{\partial^2 N_r}{\partial x \partial z} r(x,z,t)\,dxdz$$
$$= N_v v(t) + N_p \dot\phi(t) + N_r \dot\psi(t) + \sum_{i=1}^{n} [N_{\xi_i}\xi_i(t) + N_{\dot\xi_i}\dot\xi_i(t)] \quad (9)$$

The expressions for $Y_{\xi_i}$, $Y_{\dot\xi_i}$, $L_{\xi_i}$, $L_{\dot\xi_i}$, $N_{\xi_i}$, and $N_{\dot\xi_i}$ are tabulated in Appendix A.

The expression for the motion-dependent generalized force term in the $n$ elastic mode equations of motion of Eq. (1d) is given by

$$\Xi_{im}(t) = \frac{1}{\mathfrak{M}_i}\int_y \int_x \frac{\partial^2 Z_a(x,y,t)}{\partial x \partial y}\phi_i(x,y)\,dxdy + \frac{1}{\mathfrak{M}_i}\int_z \int_x \frac{\partial^2 Y(x,z,t)}{\partial x \partial z}\phi_i(x,z)\,dxdz \quad (10)$$

where $Z_a(x,y,t)$ is the antisymmetric aerodynamic force in the $z$ direction given by Eq. (11), and $Y(x,z,t)$ is the aerodynamic side force in the $y$ direction given by Eq. (12):

$$Z_a(x,y,t) = MZ_w w(x,y,t) + (I_x/y)\,[L_v v(t) + L_p\dot\phi(t) + L_r\dot\psi(t) + L_{\delta_a}\delta_a(t)] \quad (11)$$

$$Y(x,z,t) = M[Y_v v(t) + Y_p\dot\phi(t) + Y_r\dot\psi(t) + Y_{\delta_r}\delta_r(t)] + M\sum_{j=1}^{n}[Y_{\xi_j}\xi_j(t) + Y_{\dot\xi_j}\dot\xi_j(t)] \quad (12)$$

Putting Eqs. (11) and (12) into Eq. (10) gives

$$\Xi_{im}(t) = F_{i_v}v(t) + F_{i_{\dot\phi}}\dot\phi(t) + F_{i_{\dot\psi}}\dot\psi(t) + F_{i_{\delta_r}}\delta_r(t) + F_{i_{\delta_a}}\delta_a(t) + \sum_{j=1}^{n}[F_{i_{\xi_j}}\xi_j(t) + F_{i_{\dot\xi_j}}\dot\xi_j(t)] \quad (13)$$

The generalized force terms due to c.g.-referenced lateral gust velocity $v_g(t)$, rolling gust velocity $p_g(t)$, and yawing gust velocity $r_g(t)$ is given by

$$\Xi_{ig}(t) = F_{i_{v_g}}v_g(t) + F_{i_{p_g}}p_g(t) + F_{i_{r_g}}r_g(t) \quad (14)$$

$F_{i_v}$, $F_{i_{\dot\phi}}$, $F_{i_{\dot\psi}}$, $F_{i_{\delta_r}}$, $F_{i_{\delta_a}}$, $F_{i_{v_g}}$, $F_{i_{p_g}}$, $F_{i_{r_g}}$, $F_{i_{\xi_j}}$, and $F_{i_{\dot\xi_j}}$ are tabulated in Appendix A.

Substituting Eqs. (7-9, 13, and 14) into Eq. (1) performing a Laplace transformation, and putting the resulting equations in matrix form yields Eq. (15), where four antisymmetric elastic modes ($n=4$) have been explicitly included:

$$\begin{bmatrix}
s-Y_v & -Y_p s - g & (U_0 - Y_r)s & -Y_{\dot\xi_1}s - Y_{\xi_1} & -Y_{\dot\xi_2}s - Y_{\xi_2} & -Y_{\dot\xi_3}s - Y_{\xi_3} & -Y_{\dot\xi_4}s - Y_{\xi_4} \\
-L_v & s^2 - L_p s & \frac{-I_{xz}}{I_x}s^2 - L_r s & -L_{\dot\xi_1}s - L_{\xi_1} & -L_{\dot\xi_2}s - L_{\xi_2} & -L_{\dot\xi_3}s - L_{\xi_3} & -L_{\dot\xi_4}s - L_{\xi_4} \\
-N_{\dot v}s - N_v & \frac{-I_{xz}}{I_z}s^2 - N_p s & s^2 - N_r s & -N_{\dot\xi_1}s - N_{\xi_1} & -N_{\dot\xi_2}s - N_{\xi_2} & -N_{\dot\xi_3}s - N_{\xi_3} & -N_{\dot\xi_4}s - N_{\xi_4} \\
-F_{1_v} & -F_{1_{\dot\phi}}s & -F_{1_{\dot\psi}}s & s^2 + (2\zeta_1\omega_1 - F_{1_{\dot\xi_1}})s + (\omega_1^2 - F_{1_{\xi_1}}) & -F_{1_{\dot\xi_2}}s - F_{1_{\xi_2}} & -F_{1_{\dot\xi_3}}s - F_{1_{\xi_3}} & -F_{1_{\dot\xi_4}}s - F_{1_{\xi_4}} \\
-F_{2_v} & -F_{2_{\dot\phi}}s & -F_{2_{\dot\psi}}s & -F_{2_{\dot\xi_1}}s - F_{2_{\xi_1}} & s^2 + (2\zeta_2\omega_2 - F_{2_{\dot\xi_2}})s + (\omega_2^2 - F_{2_{\xi_2}}) & -F_{2_{\dot\xi_3}}s - F_{2_{\xi_3}} & -F_{2_{\dot\xi_4}}s - F_{2_{\xi_4}} \\
-F_{3_v} & -F_{3_{\dot\phi}}s & -F_{3_{\dot\psi}}s & -F_{3_{\dot\xi_1}}s - F_{3_{\xi_1}} & -F_{3_{\dot\xi_2}}s - F_{3_{\xi_2}} & s^2 + (2\zeta_3\omega_3 - F_{3_{\dot\xi_3}})s + (\omega_3^2 - F_{3_{\xi_3}}) & -F_{3_{\dot\xi_4}}s - F_{3_{\xi_4}} \\
-F_{4_v} & -F_{4_{\dot\phi}}s & -F_{4_{\dot\psi}}s & -F_{4_{\dot\xi_1}}s - F_{4_{\xi_1}} & -F_{4_{\dot\xi_2}}s - F_{4_{\xi_2}} & -F_{4_{\dot\xi_3}}s - F_{4_{\xi_3}} & s^2 + (2\zeta_4\omega_4 - F_{4_{\dot\xi_4}})s + (\omega_4^2 - F_{4_{\xi_4}})
\end{bmatrix}$$

$$\cdot \begin{bmatrix} v(s) \\ \phi(s) \\ \psi(s) \\ \xi_1(s) \\ \xi_2(s) \\ \xi_3(s) \\ \xi_4(s) \end{bmatrix} = \begin{bmatrix} Y_{\delta_a} & Y_{\delta_r} \\ L_{\delta_a} & L_{\delta_r} \\ N_{\delta_a} & N_{\delta_r} \\ F_{1_{\delta_a}} & F_{1_{\delta_r}} \\ F_{2_{\delta_a}} & F_{2_{\delta_r}} \\ F_{3_{\delta_a}} & F_{3_{\delta_r}} \\ F_{4_{\delta_a}} & F_{4_{\delta_r}} \end{bmatrix} \begin{bmatrix} \delta_a(s) \\ \delta_r(s) \end{bmatrix} + \begin{bmatrix} Y_v & Y_p & Y_r \\ L_v & L_p & L_r \\ N_v & N_p & N_r \\ F_{1_{v_g}} & F_{1_{p_g}} & F_{1_{r_g}} \\ F_{2_{v_g}} & F_{2_{p_g}} & F_{2_{r_g}} \\ F_{3_{v_g}} & F_{3_{p_g}} & F_{3_{r_g}} \\ F_{4_{v_g}} & F_{4_{p_g}} & F_{4_{r_g}} \end{bmatrix} \begin{bmatrix} v_g(s) \\ p_g(s) \\ r_g(s) \end{bmatrix} \quad (15)$$

## Stability Derivatives Evaluation

Elastic mode shape and slope data are usually given as a function of lumped mass stations, which in our formulation, are given in $xy$ and $xz$ coordinates. The double integral terms of Appendix A can be approximated as summations over incremental areas $\Delta x\Delta y$ or $\Delta x\Delta z$ associated with each mass point. It is necessary to develop methods for evaluating the partial derivatives at each point. For example, using Eq. (2),

$$\frac{\partial^2 N_v}{\partial x \partial z} = \frac{\rho U_0 S b}{2I_z} \frac{\partial^2 C_{n_\beta}}{\partial x \partial z} \tag{16}$$

The total-airplane rigid-body nondimensional stability derivatives, such as $C_{n_\beta}$, are known constants for a trim flight condition. However, the $xy$ or $xz$ area distributions of these are required.

As an example, the distribution for $C_{n_\beta}$ is presented to illustrate the method used in our formulation. Details for the evaluation of the other stability derivative distributions are given in Appendix B.

The distribution for $C_{n_\beta}$ can be approximated in terms of the fuselage side profile area $S_{fa}$ ahead of the c.g., $S_{fb}$ behind the c.g., and vertical tail area $S_{VT}$. For a directionally statically stable airplane, $C_{n_\beta}$ is positive; the following expressions are valid only for this condition:

$$C_{n_\beta} = C_{n_{\beta_{fa}}} + (C_{n_{\beta_{fb}}} + C_{n_{\beta_{VT}}}) \tag{17}$$

$C_{n_{\beta_{fa}}}$ is negative and $C_{n_{\beta_{fb}}}$ and $C_{n_{\beta_{VT}}}$ are positive. The component values in Eq. (17) are reasonably approximated by

$$C_{n_{\beta_{fa}}} = -C_{n_\beta} S_{fa} / (S_{fb} + S_{VT}) \tag{18}$$

$$(C_{n_{\beta_{fb}}} + C_{n_{\beta_{VT}}}) = C_{n_\beta} [S_{fa} + S_{fb} + S_{VT}] / [S_{fb} + S_{VT}] \tag{19}$$

$$C_{n_{\beta_{VT}}} = (C_{n_{\beta_{fb}}} + C_{n_{\beta_{VT}}}) S_{VT} / [S_{fb} + S_{VT}] \tag{20}$$

At the c.g., $(\partial^2 C_{n_\beta} / \partial x \partial z) = 0$. Thus, each component of $C_{n_\beta}$ can be approximated as a linear variation with $x$ away from the c.g., such as in Fig. 3.

The analytical expressions are

$$\begin{aligned}
\frac{\partial^2 C_{n_\beta}}{\partial x \partial z} &= \frac{2x}{x_a} \frac{C_{n_{\beta_{fa}}}}{S_{fa}} && \text{for } 0 \le x \le x_a \\
&= \frac{2x}{x_T} \frac{C_{n_{\beta_{fb}}}}{S_{fb}} && \text{for } x_b \le x \le 0 \\
&= \frac{2x}{x_T} \frac{C_{n_{\beta_{fb}}}}{S_{fb}} + \frac{2(x-x_b)}{(x_T - x_b)} \frac{C_{n_{\beta_{VT}}}}{S_{VT}} && \text{for } x_T \le x \le x_b \\
&= 0 && \text{for all } y \neq 0
\end{aligned} \tag{21}$$

With the airplane elastic mode shapes and slopes and the rigid-body total-airplane stability derivatives, all the terms in Appendix A, and thus the coefficients in Eq. (15), can be computed.

## Verification with B-1 Airplane Dynamics

To verify the accuracy of this method, the terms in Eq. (15) were calculated for the B-1 at a sea-level, Mach 0.85 flight condition and compared with the corresponding terms in equations that were generated by other methods.[6,7] Approximately 80% of the terms showed reasonable agreement with the B-1 data. In view of the highly approximate nature of the formulation, this is encouraging evidence for the usefulness of the method.[8]

**Table 1 Frequencies and damping ratios**

| B-1 Data | | Present method | |
|---|---|---|---|
| Frequency | Damping | Frequency | Damping |
| 1.9114 | 0.078124 | 2.0931 | 0.09523 |
| 15.6095 | 0.036444 | 15.7100 | 0.10310 |
| 16.5051 | 0.118676 | 16.1920 | 0.66130 |
| 20.2896 | 0.014355 | 20.2342 | 0.02514 |
| 24.1421 | 0.155250 | 29.5215 | 0.17633 |

A further check was made by comparing the roots of the characteristic equations for the B-1 data and this formulation by expanding the determinant of the $7 \times 7$ matrix of polynomials and coefficients in Eq. (15). The coupled undamped natural frequencies in rad/s and damping ratios were calculated for each pair of complex roots. The comparisons are shown in Table 1. The agreement is better with frequencies than with damping ratios, with the second mode showing the largest difference in damping. The four antisymmetric elastic modes of the B-1 had free–free undamped natural frequencies of 12.932, 16.4896, 20.1721, and 22.1369 rad/s. All had structural damping ratios of 0.02. The first line of numbers in Table 1 corresponds to the dutch-roll frequency and damping ratio.

## Conclusions

This method of formulation of the lateral-directional small perturbation equations of motion for elastic airplanes allows the expression of aerodynamic forces and moments due to elastic vibration in terms of rigid-body aerodynamic stability derivatives. Thus, it is potentially a useful and powerful tool in the preliminary phase of flexible aircraft stability and control, handling and ride qualities, and control system design studies.

The accuracy of this method was established for the longitudinal dynamics in Ref. 3 and has been verified in the current work for the lateral-directional dynamics by comparison to more accurate data for the B-1 airplane. The lack of complete information on the planform geometry of the B-1 and our having to calculate mode slopes analytically by a linear approximation from mode shape data may account for some of the discrepancies. Therefore, the new method is possibly more accurate than this example indicates.

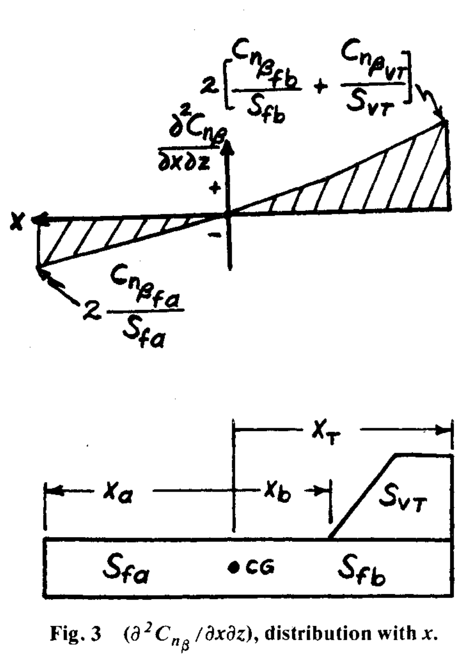

Fig. 3 $(\partial^2 C_{n_\beta} / \partial x \partial z)$, distribution with $x$.

## Appendix A: Equation (15) Coefficients

$$Y_{\xi_i} = -U_0 \int_z \int_x \frac{\partial^2 Y_v}{\partial x \partial z} \phi_i'(x,z)\,\mathrm{d}x\mathrm{d}z \tag{A1}$$

$$Y_{\dot{\xi}_i} = \int_z \int_x \left[ \frac{\partial^2 Y_v}{\partial x \partial z} \phi_i(x,z) + \frac{\partial^2 Y_r}{\partial x \partial z} \phi_i'(x,z) \right] \mathrm{d}x\mathrm{d}z \tag{A2}$$

$$L_{\xi_i} = -U_0 \int_z \int_x \frac{\partial^2 L_v}{\partial x \partial z} \phi_i'(x,z)\,\mathrm{d}x\mathrm{d}z - \frac{U_0 M}{I_x} \int_y \int_x y \frac{\partial^2 Z_w}{\partial x \partial y} \phi_i'(x,y)\,\mathrm{d}x\mathrm{d}y \tag{A3}$$

$$L_{\dot{\xi}_i} = \int_z \int_x \left[ \frac{\partial^2 L_v}{\partial x \partial z} \phi_i(x,z) + \frac{\partial^2 L_r}{\partial x \partial z} \phi_i'(x,z) \right] \mathrm{d}x\mathrm{d}z + \int_y \int_x \frac{\partial^2 L_p}{\partial x \partial y}{}' \phi_i(x,y)\,\mathrm{d}x\mathrm{d}y + \frac{M}{I_x} \int_y \int_x y \frac{\partial^2 Z_w}{\partial x \partial y} \phi_i(x,y)\,\mathrm{d}x\mathrm{d}y \tag{A4}$$

$$N_{\xi_i} = -U_0 \int_z \int_x \frac{\partial^2 N_v}{\partial x \partial z} \phi_i'(x,z)\,\mathrm{d}x\mathrm{d}z \tag{A5}$$

$$N_{\dot{\xi}_i} = \int_z \int_x \left[ \frac{\partial^2 N_v}{\partial x \partial z} \phi_i(x,z) + \frac{\partial^2 N_r}{\partial x \partial z} \phi_i'(x,z) \right] \mathrm{d}x\mathrm{d}z + \int_y \int_x \frac{\partial^2 N_p}{\partial x \partial y}{}' \phi_i(x,y)\,\mathrm{d}x\mathrm{d}y \tag{A6}$$

$$F_{i_v} = \frac{M}{\mathfrak{M}_i} \int_z \int_x \frac{\partial^2 Y_v}{\partial x \partial z} \phi_i(x,z)\,\mathrm{d}x\mathrm{d}z + \frac{I_x}{\mathfrak{M}_i} \int_y \int_x \frac{1}{y} \frac{\partial^2 L_v}{\partial x \partial y} \phi_i(x,y)\,\mathrm{d}x\mathrm{d}y = F_{i_{v_g}} \tag{A7}$$

$$F_{i_{\dot{\phi}}} = \frac{M}{\mathfrak{M}_i} \int_z \int_x \frac{\partial^2 Y_p}{\partial x \partial z} \phi_i(x,z)\,\mathrm{d}x\mathrm{d}z + \frac{I_x}{\mathfrak{M}_i} \int_y \int_x \frac{1}{y} \frac{\partial^2 L_p}{\partial x \partial y} \phi_i(x,y)\,\mathrm{d}x\mathrm{d}y = F_{i_{p_g}} \tag{A8}$$

$$F_{i_{\dot{\psi}}} = \frac{M}{\mathfrak{M}_i} \int_z \int_x \frac{\partial^2 Y_r}{\partial x \partial z} \phi_i(x,z)\,\mathrm{d}x\mathrm{d}z + \frac{I_x}{\mathfrak{M}_i} \int_y \int_x \frac{1}{y} \frac{\partial^2 L_r}{\partial x \partial y} \phi_i(x,y)\,\mathrm{d}x\mathrm{d}y = F_{i_{r_g}} \tag{A9}$$

$$F_{i_{\dot{\xi}_j}} = \frac{M}{\mathfrak{M}_i} \int_y \int_x \frac{\partial^2 Z_w}{\partial x \partial y} \phi_i(x,y) \phi_j(x,y)\,\mathrm{d}x\mathrm{d}y + \frac{M}{\mathfrak{M}_i} \int_z \int_x \left[ \frac{\partial^2 Y_v}{\partial x \partial z} \phi_j(x,z) + \frac{\partial^2 Y_r}{\partial x \partial z} \phi_j'(x,z) \right] \phi_i(x,z)\,\mathrm{d}x\mathrm{d}z \tag{A10}$$

$$F_{i_{\xi_j}} = -\frac{U_0 M}{\mathfrak{M}_i} \int_y \int_x \frac{\partial^2 Z_w}{\partial x \partial y} \phi_i(x,y) \phi_j'(x,y)\,\mathrm{d}x\mathrm{d}y - \frac{U_0 M}{\mathfrak{M}_i} \int_z \int_x \frac{\partial^2 Y_v}{\partial x \partial z} \phi_i(x,z) \phi_j'(x,z)\,\mathrm{d}x\mathrm{d}z \tag{A11}$$

$$F_{i_{\delta_r}} = \frac{M}{\mathfrak{M}_i} \int_z \int_x \frac{\partial^2 Y_{\delta_r}}{\partial x \partial z} \phi_i(x,z)\,\mathrm{d}x\mathrm{d}z \tag{A12}$$

$$F_{i_{\delta_a}} = \frac{I_x}{\mathfrak{M}_i} \int_y \int_x \frac{1}{y} \frac{\partial^2 L_{\delta_a}}{\partial x \partial y} \phi_i(x,y)\,\mathrm{d}x\mathrm{d}y \tag{A13}$$

## Appendix B: Stability Derivative Distributions

The fuselage and vertical tail side profile are the primary contributors to $C_{y_\beta}$. In terms of the fuselage and vertical tail reference areas, the portion of $C_{y_\beta}$ due to the vertical tail and the fuselage can be approximated as

$$C_{y_{\beta_{VT}}} = C_{y_\beta} S_{VT} / (S_f + S_{VT}) \tag{B1}$$

$$C_{y_{\beta_f}} = C_{y_\beta} - C_{y_{\beta_{VT}}} \tag{B2}$$

Therefore,

$$\begin{aligned} \partial^2 C_{y_\beta} / \partial x \partial z &= C_{y_{\beta_f}} / S_f && \text{for fuselage stations } x,\ z \\ &= C_{y_{\beta_{VT}}} / S_{VT} && \text{for vertical tail stations } x,\ z \\ &= 0 && \text{elsewhere} \end{aligned} \tag{B3}$$

The $C_{y_p}$ distribution is due mainly to the vertical tail, with the fuselage assumed to contribute a negligible amount. Thus,

$$\begin{aligned} \partial^2 C_{y_p} / \partial x \partial z &= C_{y_{p_{VT}}} / S_{VT} && \text{for vertical tail stations } x,\ z \\ &= 0 && \text{elsewhere} \end{aligned} \tag{B4}$$

The fuselage and vertical tail are the primary contributors to $C_{y_r}$. The development of the $C_{y_r}$ distribution is analogous to that for $C_{n_\beta}$. Thus,

$$C_{y_r} = C_{y_{r_{fa}}} + C_{y_{r_{VT}}} + C_{y_{r_{fb}}} \tag{B5}$$

$$C_{y_{r_{fa}}} = -C_{y_r} S_{fa} / (S_{fb} + S_{VT}) \tag{B6}$$

$$(C_{y_{r_{fb}}} + C_{y_{r_{VT}}}) = C_{y_r} [S_{fa} + S_{fb} + S_{VT}] / [S_{fb} + S_{VT}] \tag{B7}$$

$$C_{y_{r_{VT}}} = (C_{y_{r_{fb}}} + C_{y_{r_{VT}}}) S_{VT} / [S_{fb} + S_{VT}] \tag{B8}$$

These are valid only for $C_{y_r} > 0$.

The linear distribution of $C_{y_r}$, as shown in Fig. 4, is

$$\begin{aligned} \frac{\partial^2 C_{y_r}}{\partial x \partial z} &= \frac{2x}{x_a} \frac{C_{y_{r_{fa}}}}{S_{fa}} && \text{for } 0 \le x \le x_a \\ &= \frac{2x}{x_T} \frac{C_{y_{r_{fb}}}}{S_{fb}} && \text{for } x_b \le x \le 0 \\ &= \frac{2x}{x_T} \frac{C_{y_{r_{fb}}}}{S_{fb}} + \frac{2(x - x_b)}{(x_T - x_b)} \frac{C_{y_{r_{VT}}}}{S_{VT}} && \text{for } x_T \le x \le x_b \\ &= 0 && \text{for all } y \neq 0 \end{aligned} \tag{B9}$$

Since $C_{y_{\delta_r}}$ is due only to the vertical tail area,

$$\begin{aligned} \partial^2 C_{y_{\delta_r}} / \partial x \partial z &= C_{y_{\delta_r}} / S_{VT} && \text{for vertical tail stations } x,\ z \\ &= 0 && \text{elsewhere} \end{aligned} \tag{B10}$$

The wing and vertical tail contribute the majority of $C_{l_\beta}$.

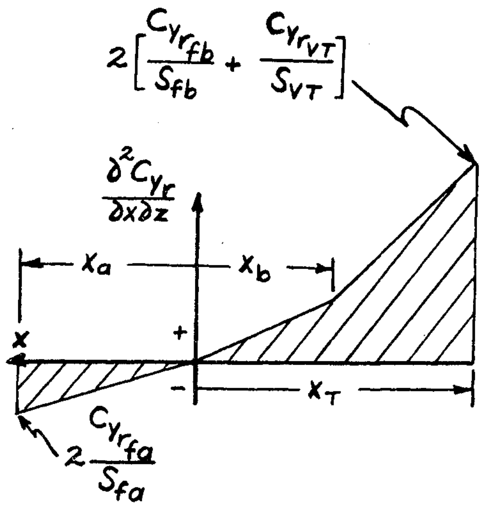


Fig. 4 $(\partial^2 C_{y_r}/\partial x \partial z)$ distribution with $x$.

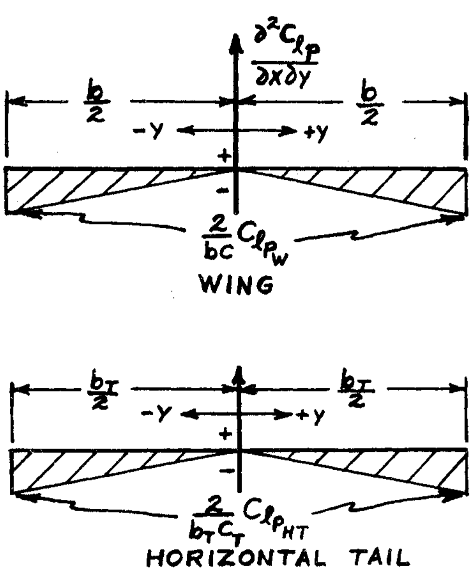


Fig. 5 $(\partial^2 C_{l_p}/\partial x \partial y)$, distribution with $y$.

The $xz$ distribution is due to the vertical tail. Therefore,

$$C_{l_{\beta_{VT}}} = C_{l_\beta} S_{VT} / (S_W + S_{VT}) \tag{B11}$$

Thus,

$$\begin{aligned} \partial^2 C_{l_\beta} / \partial x \partial z &= C_{l_{\beta_{VT}}} / S_{VT} && \text{for vertical tail stations } x, z \\ &= 0 && \text{elsewhere} \end{aligned} \tag{B12}$$

The $xy$ distribution is due to the wing with a negligible amount from the horizontal tail.

$$C_{l_{\beta_W}} = C_{l_\beta} S_W / (S_W + S_{VT}) \tag{B13}$$

Thus,

$$\begin{aligned} \partial^2 C_{l_\beta} / \partial x \partial y &= C_{l_{\beta_W}} / S_w && \text{for wing stations } x, y \\ &= 0 && \text{elsewhere} \end{aligned} \tag{B14}$$

The major contribution to a negative valued $C_{l_p}$ is from the wing, with a smaller amount from the horizontal tail, and a negligible contribution from the vertical tail.

$$C_{l_{p_W}} = C_{l_p} S_W / (S_W + S_{HT}) \tag{B15}$$

$$C_{l_{p_{HT}}} = C_{l_p} S_{HT} / (S_W + S_{HT}) \tag{B16}$$

The approximation to $(\partial^2 C_{l_p} / \partial x \partial y)$ will be made one variable at a time. Taking $c$ as the mean aerodynamic wing chord, $(\partial C_{l_{p_W}} / \partial x)$ is taken as constant across the wing chord, with a value given by

$$\partial C_{l_{p_W}} / \partial x = C_{l_{p_W}} / c \tag{B17}$$

Likewise, for the horizontal tail, with $c_t$ as its mean chord,

$$\partial C_{l_{p_{HT}}} / \partial x = C_{l_{p_{HT}}} / c_t \tag{B18}$$

The $y$ distribution is taken as linear across the wing or tail span, as indicated in Fig. 5. Thus,

$$\begin{aligned} \frac{\partial^2 C_{l_p}}{\partial x \partial y} &= \frac{4|y|}{b^2 c} C_{l_{p_W}} && \text{for wing stations } x, y \\ &= \frac{4|y|}{b_t^2 c_t} C_{l_{p_{HT}}} && \text{for horizontal tail stations } x, y \\ &= 0 && \text{elsewhere} \end{aligned} \tag{B19}$$

The wing and vertical tail are the major contributors to the total $C_{l_r}$, with a negligible contribution from the horizontal tail. The distribution in the $xz$ plane is due only to the vertical tail. Since the distance of the vertical tail center of pressure above the $x$ axis (negative $z$ direction) is relatively small, the distribution is assumed constant with $z$, rather than linear. Therefore, for the vertical tail,

$$C_{l_{r_{VT}}} = C_{l_r} S_{VT} / (S_W + S_{VT}) \tag{B20}$$

Thus,

$$\begin{aligned} \partial^2 C_{l_r} / \partial x \partial z &= C_{l_{r_{VT}}} && \text{for vertical tail stations } x, z \\ &= 0 && \text{elsewhere} \end{aligned} \tag{B21}$$

The $xy$ distribution is due to the wing.

$$C_{l_{r_W}} = C_{l_r} S_W / (S_W + S_{VT}) \tag{B22}$$

Thus,

$$\begin{aligned} \partial^2 C_{l_r} / \partial x \partial y &= C_{l_{r_W}} / S_W && \text{for wing stations } x, y \\ &= 0 && \text{elsewhere} \end{aligned} \tag{B23}$$

By letting $S_a$ represent the planform area of both ailerons or the horizontal tail area for airplanes that use differential stabilator deflection for roll control, then an approximation is

$$\begin{aligned} \partial^2 C_{l_{\delta_a}} / \partial x \partial y &= C_{l_{\delta_a}} / S_a && \text{for aileron stations } x, y \\ &= 0 && \text{elsewhere} \end{aligned} \tag{B24}$$

The wing and vertical tail are the main contributors to $C_{n_p}$. $C_{n_{p_{VT}}}$ is nearly always positive, however $C_{n_{p_W}}$ can be either positive or negative. Therefore, let

$$C_{n_p} = C_{n_{p_W}} + C_{n_{p_{VT}}} \tag{B25}$$

An approximation to $C_{n_{p_{VT}}}$ is given in Ref. 9 as

$$C_{n_{p_{VT}}} = (-2/b^2) [l_{VT} Z_{VT} - l_{VT}^2 \alpha] C_{y_{\beta_{VT}}} \tag{B26}$$

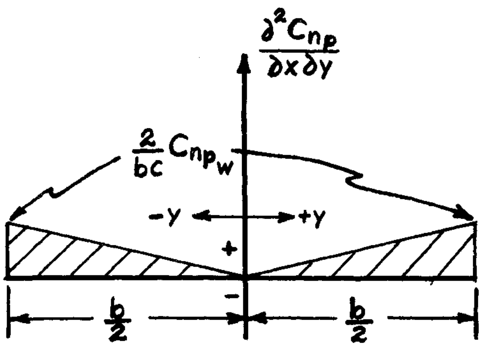


Fig. 6 $(\partial^2 C_{n_p}/\partial x \partial y)$ distribution with $y$.

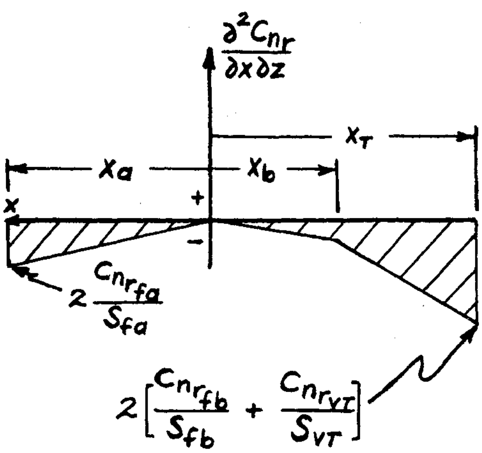


Fig. 7 $(\partial^2 C_{n_r}/\partial x \partial z)$ distribution with $x$.

where $C_{y_{\beta_{VT}}}$ is given by Eq. (1B), $l_{VT}$ is the distance from the airplane c.g. to the vertical tail aerodynamic center, $Z_{VT}$ is the distance from the $x$ axis to the vertical tail aerodynamic center, and $\alpha$ is the trim angle of attack in radians. Knowing $C_{n_p}$, then

$$C_{n_{p_W}} = C_{n_p} - C_{n_{p_{VT}}} \tag{B27}$$

Take $(\partial C_{n_{p_W}}/\partial x)$ as constant across the chord, with a value given by

$$\partial C_{n_{p_W}}/\partial x = C_{n_{p_W}}/c \tag{B28}$$

The $y$ distribution is taken as linear, as shown in Fig. 6. Thus,

$$\frac{\partial^2 C_{n_p}}{\partial x \partial y} = \frac{4|y|}{b^2 c} C_{n_{p_W}} \quad \text{for wing stations } x, y$$
$$= 0 \quad \text{elsewhere} \tag{B29}$$

The contributions to $C_{n_r}$ come mainly from the fuselage and vertical tail:

$$C_{n_r} = C_{n_{r_{fa}}} + C_{n_{r_{fb}}} + C_{n_{r_{VT}}} \tag{B30}$$

Each of the terms in Eq. (B30) is negative. Based on profile areas, we have

$$C_{n_{r_{fa}}} = C_{n_r} S_{fa}/(S_{fa} + S_{fb} + S_{VT}) \tag{B31}$$

$$C_{n_{r_{fb}}} = C_{n_r} S_{fb}/(S_{fa} + S_{fb} + S_{VT}) \tag{B32}$$

$$C_{n_{r_{VT}}} = C_{n_r} S_{VT}/(S_{fa} + S_{fb} + S_{VT}) \tag{B33}$$

At the c.g., $(\partial^2 C_{n_r}/\partial x \partial z) = 0$. Thus, each of the components of $C_{n_r}$ can be approximated as a linear variation with $x$ away from the c.g., such as in Fig. 7. So,

$$\frac{\partial^2 C_{n_r}}{\partial x \partial z} = \frac{2x}{x_a} \frac{C_{n_{r_{fa}}}}{S_{fa}} \quad \text{for } 0 \le x \le x_a$$
$$= \frac{2x}{x_T} \frac{C_{n_{r_{fb}}}}{S_{fb}} \quad \text{for } x_b \le x \le 0$$
$$= \frac{2x}{x_T} \frac{C_{n_{r_{fb}}}}{S_{fb}} + \frac{2(x - x_b)}{(x_T - x_b)} \frac{C_{n_{r_{VT}}}}{S_{VT}} \quad \text{for } x_T \le x \le x_b$$
$$= 0 \quad \text{for all } y \ne 0 \tag{B34}$$

For conventional-tailed airplanes, the lift curve slope can be reasonably approximated by

$$C_{L_\alpha} = C_{L_{\alpha_W}} + C_{L_{\alpha_{HT}}} \tag{B35}$$

where $C_{L_{\alpha_W}}$ and $C_{L_{\alpha_{HT}}}$ are the wing and horizontal tail contributions. Fuselage lift is considered to be small and is neglected. Methods for computing these can be found in Ref. 2. The horizontal tail contribution is approximately 10% of the total. Therefore, following the procedures of Ref. 3,

$$\partial^2 C_{L_\alpha}/\partial x \partial y = 0 \quad \text{for fuselage stations } x(y=0)$$
$$= (0.9/S_w) C_{L_\alpha} \quad \text{for wing stations } x, y$$
$$= (0.1/S_{HT}) C_{L_\alpha} \quad \text{for horizontal tail stations } x, y$$


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